Lecture 2 answers to exercises: Matrices

1. (*a.*) $\begin{bmatrix} -10 & 8 \\ 5 & 7 \end{bmatrix}$ (*b.*) $\begin{bmatrix} -1 \end{bmatrix}$ (*c.*) $\begin{bmatrix} -5 & 0 & 10 \\ -1 & 0 & 2 \\ -3 & 0 & 6 \end{bmatrix}$

2. $\begin{bmatrix} -10 & 8 & 14 \\ 6 & 8 & 5 \end{bmatrix}$

3. Yes. cA is the matrix A where every entry is multiplied by c, so its entries are ca_{ij}. Therefore, $D = (cA)B$ has as its entries $d_{ij} = \sum_{k=1}^{n} ca_{ik}b_{kj}$.

   Similarly, the matrix $E = c(AB)$ has as its entries $e_{ij} = c \cdot \sum_{k=1}^{n} a_{ik}b_{kj}$. It is now easy to see that $d_{ij} = e_{ij}$, and hence the matrices D and E are the same.

4. Yes. This can be seen from the formula $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ which gives the values of the matrix that is the multiplication. If all a_{ik} are zero, or all b_{kj} are zero, then all c_{ij} will be zero as well.

   No. Here is a counterexample:

   $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5. $$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

6. $$\left[\begin{array}{cc|c} 6 & 4 & 9 \\ 3 & 5 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 0 & -6 & 13 \\ 3 & 5 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 0 & 1 & -\frac{13}{6} \\ 3 & 5 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 0 & 1 & -\frac{13}{6} \\ 3 & 0 & \frac{53}{6} \end{array}\right] \rightsquigarrow$$

   $$\left[\begin{array}{cc|c} 0 & 1 & -\frac{13}{6} \\ 1 & 0 & \frac{53}{18} \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{53}{18} \\ 0 & 1 & -\frac{13}{6} \end{array}\right]$$

   So the system solves to $x = \frac{53}{18} = 2\frac{17}{18}$ and $y = -\frac{13}{6} = -2\frac{1}{6}$.

7. $$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 4 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 2 & -1 & 1 & 3 \\ 0 & 3 & 2 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & -3 & 5 & -1 \\ 0 & 3 & 2 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 0 & 7 & 6 \\ 0 & 3 & 2 & 7 \end{array}\right] \rightsquigarrow$$

   $$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & \frac{6}{7} \\ 0 & 3 & 2 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & \frac{6}{7} \\ 0 & 3 & 0 & \frac{37}{7} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{37}{21} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{26}{7} \\ 0 & 0 & 1 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{37}{21} \end{array}\right] \rightsquigarrow$$

   $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{41}{21} \\ 0 & 0 & 1 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{37}{21} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{41}{21} \\ 0 & 1 & 0 & \frac{37}{21} \\ 0 & 0 & 1 & \frac{6}{7} \end{array}\right]$$

   So the system solves to $x = \frac{41}{21}$ and $y = \frac{37}{21}$ and $z = \frac{6}{7}$.

8. We notice that we now have three equations, but only two unknowns to try to satisfy these equations. Geometrically, we have three implicit equations of lines in the plane, and we wonder if there is any point that lies on all three lines simultaneously.

   We can still use Gaussian elimination to find the solution, if it exist:

$$\left[\begin{array}{rr|r} 7 & 4 & 12 \\ 3 & 5 & -2 \\ 1 & -1 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|r} 7 & 4 & 12 \\ 0 & 8 & -14 \\ 1 & -1 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|r} 7 & 4 & 12 \\ 0 & 1 & -14/8 \\ 1 & -1 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|r} 7 & 4 & 12 \\ 0 & 1 & -14/8 \\ 1 & 0 & 18/8 \end{array}\right]$$

   Looking at the last augmented matrix, we see that the values for x and y are already known in the bottom two rows. We can test by substitution whether these values satisfy the top equation as well:

$$7x + 4y = 7 \cdot \frac{18}{8} + 4 \cdot \frac{-14}{8} = \frac{35}{4} .$$

   Since this is not equal to 12, we conclude that the system does not have a solution.

   If we would have continued with Gaussian elimination, we would have got a top row $0 \;\; 0 \mid 3\frac{1}{4}$, which states that $0 \cdot x + 0 \cdot y = 3\frac{1}{4}$, which is never true of course, and hence we again see that the system of equations does not have a solution. If the top row would have been $0 \;\; 0 \mid 0$, then the system would have had a solution.

9. When you apply Gaussian elimination, you will notice that one of the rows becomes $0 \; 0 \; 0 \mid 0$, so this equation is redundant (we can throw it away; it is always satisfied). The two remaining equations still have three unknowns, and there will be many solutions to the system.

   Geometrically, the three planes given by their implicit representations intersect in one and the same line.

   If the last equation should be equal to 3 instead of 4, Gaussian elimination will eventually give a row that looks like $0 \;\; 0 \;\; 0 \mid c$ with c some value not zero. Now the system has no solutions. Geometrically, the three planes intersect pairwise in three lines, and these lines are parallel to each other.

10.

$$\left[\begin{array}{rr|rr} 2 & 3 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|rr} 0 & 11 & 1 & 2 \\ -1 & 4 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|rr} 0 & 1 & 1/11 & 2/11 \\ -1 & 4 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{rr|rr} 0 & 1 & 1/11 & 2/11 \\ -1 & 0 & -4/11 & 3/11 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|rr} 0 & 1 & 1/11 & 2/11 \\ 1 & 0 & 4/11 & -3/11 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|rr} 1 & 0 & 4/11 & -3/11 \\ 0 & 1 & 1/11 & 2/11 \end{array}\right]$$

   The inverse matrix is $\begin{bmatrix} 4/11 & -3/11 \\ 1/11 & 2/11 \end{bmatrix}$

   To verify that the answer is correct, we multiply the original matrix with the computed inverse, and see if we get the identity matrix.

11.

$$\left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 7 & 2 & -3 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 1 & -7 & 0 & -3 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -3 & 1 & 1 \\ 1 & -7 & 0 & -3 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & -7 & 0 & -3 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{rrr|rrr} 2 & 0 & -1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & -7 & 0 & -3 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rrr} 2 & 0 & -1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 0 & 0 & 4 & -2\frac{1}{3} & -1\frac{1}{3} \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{rrr|rrr} 0 & 0 & -1 & -10 & 5\frac{2}{3} & 3\frac{2}{3} \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 0 & 0 & 4 & -2\frac{1}{3} & -1\frac{1}{3} \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rrr} 0 & 0 & 1 & 10 & -5\frac{2}{3} & -3\frac{2}{3} \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 0 & 0 & 4 & -2\frac{1}{3} & -1\frac{1}{3} \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 4 & -2\frac{1}{3} & -1\frac{1}{3} \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 10 & -5\frac{2}{3} & -3\frac{2}{3} \end{array}\right]$$

So the inverse matrix is

$$\left[\begin{array}{rrr} 4 & -2\frac{1}{3} & -1\frac{1}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{3} \\ 10 & -5\frac{2}{3} & 2 \end{array}\right]$$

12. Yes. The matrix (3) for instance, can even be inverted using the usual Gaussian elimination. It looks a bit strange, but we can simply write: $(3 \mid 1) \rightsquigarrow (1 \mid \frac{1}{3})$, so the inverse matrix is $(\frac{1}{3})$. Multiplication of the matrices (3) and $(\frac{1}{3})$ gives the identity matrix (1).